Do not be scared by the mathematical details. Follow the compression and decompression process considering the parts highlighted in yellow.

Image Compression and the Discrete Cosine Transform

Ken Cabeen and Peter Gent

Math 45
College of the Redwoods

Abstract. The mathematical equations of the DCT and its uses with image compression are explained.

Introduction

As our use of and reliance on computers continues to grow, so too does our need for efficient ways of storing large amounts of data. For example, someone with a web page or online catalog – that uses dozens or perhaps hundreds of images – will more than likely need to use some form of image compression to store those images. This is because the amount of space required to hold unadulterated images can be prohibitively large in terms of cost. Fortunately, there are several methods of image compression available today. These fall into two general categories: lossless and lossy image compression. The JPEG process is a widely used form of lossy image compression that centers around the Discrete Cosine Transform. The DCT works by separating images into parts of differing frequencies. During a step called quantization, where part of compression actually occurs, the less important frequencies are discarded, hence the use of the term "lossy." Then, only the most important frequencies that remain are used retrieve the image in the decompression process. As a result, reconstructed images contain some distortion; but as we shall soon see, these levels of distortion can be adjusted during the compression stage. The JPEG method is used for both color and black-and-white images, but the focus of this article will be on compression of the latter.

The Process

The following is a general overview of the JPEG process. Later, we will take the reader through a detailed tour of JPEG's method so that a more comprehensive understanding of the process may be acquired.

1. The image is broken into 8x8 blocks of pixels.
2. Working from left to right, top to bottom, the DCT is applied to each block.
3. Each block is compressed through quantization.
4. The array of compressed blocks that constitute the image is stored in a drastically reduced amount of space.
5. When desired, the image is reconstructed through decompression, a process that uses the Inverse Discrete Cosine Transform (IDCT).

The DCT Equation

The DCT equation (Eq. 1) computes the i,jth entry of the DCT of an image.

$$D(i,j) = \frac{1}{\sqrt{2N}} C(i)C(j) \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} p(x,y) \cos\left[\frac{(2x+1)i\pi}{2N} \right] \cos\left[\frac{(2y+1)j\pi}{2N} \right] \qquad 1$$

$$C(u) = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } u = 0 \\ 1 & \text{if } u > 0 \end{cases} \qquad 2$$

$p(x,y)$ is the x,yth element of the image represented by the matrix p. N is the size of the block that the DCT is done on. The equation calculates one entry (i,jth) of the transformed image from the pixel values of the original image matrix. For the standard 8x8 block that JPEG compression uses, N equals 8 and x and y range from 0 to 7. Therefore $D(i,j)$ would be as in Equation (3).

$$D(i,j) = \frac{1}{4}C(i)C(j)\sum_{x=0}^{7}\sum_{y=0}^{7}p(x,y)\cos\left[\frac{(2x+1)i\pi}{16}\right]\cos\left[\frac{(2y+1)j\pi}{16}\right] \qquad 3$$

Because the DCT uses cosine functions, the resulting matrix depends on the horizontal, diagonal, and vertical frequencies. Therefore an image black with a lot of change in frequency has a very random looking resulting matrix, while an image matrix of just one color, has a resulting matrix of a large value for the first element and zeroes for the other elements.

The DCT Matrix

To get the matrix form of Equation (1), we will use the following equation

$$T_{i,j} = \begin{cases} \frac{1}{\sqrt{N}} & \text{if } i = 0 \\ \sqrt{\frac{2}{N}}\cos\left[\frac{(2j+1)i\pi}{2N}\right] & \text{if } i > 0 \end{cases} \qquad 4$$

For an 8x8 block it results in this matrix:

$$T = \begin{bmatrix} .3536 & .3536 & .3536 & .3536 & .3536 & .3536 & .3536 & .3536 \\ .4904 & .4157 & .2778 & .0975 & -.0975 & -.2778 & -.4157 & -.4904 \\ .4619 & .1913 & -.1913 & -.4619 & -.4619 & -.1913 & .1913 & .4619 \\ .4157 & -.0975 & -.4904 & -.2778 & .2778 & .4904 & .0975 & -.4157 \\ .3536 & -.3536 & -.3536 & .3536 & .3536 & -.3536 & -.3536 & .3536 \\ .2778 & -.4904 & .0975 & .4157 & -.4157 & -.0975 & .4904 & -.2778 \\ .1913 & -.4619 & .4619 & -.1913 & -.1913 & .4619 & -.4619 & .1913 \\ .0975 & -.2778 & .4157 & -.4904 & .4904 & -.4157 & .2778 & -.0975 \end{bmatrix}$$

The first row ($i = 1$) of the matrix has all the entries equal to $1/\sqrt{8}$ as expected from Equation (4).

The columns of T form an orthonormal set, so T is an orthogonal matrix. When doing the inverse DCT the orthogonality of T is important, as the inverse of T is T' which is easy to calculate.

Doing the DCT on an 8x8 Block

Before we begin, it should be noted that the pixel values of a black-and-white image range from 0 to 255 in steps of 1, where pure black is represented by 0, and pure white by 255. Thus it can be seen how a photo, illustration, etc. can be accurately represented by these 256 shades of gray.

Since an image comprises hundreds or even thousands of 8x8 blocks of pixels, the following description of what happens to one 8x8 block is a microcosm of the JPEG process;

what is done to one block of image pixels is done to all of them, in the order earlier specified.

Now, let's start with a block of image-pixel values. This particular block was chosen from the very upper- left-hand corner of an image.

$$Original = \begin{bmatrix} 154 & 123 & 123 & 123 & 123 & 123 & 123 & 136 \\ 192 & 180 & 136 & 154 & 154 & 154 & 136 & 110 \\ 254 & 198 & 154 & 154 & 180 & 154 & 123 & 123 \\ 239 & 180 & 136 & 180 & 180 & 166 & 123 & 123 \\ 180 & 154 & 136 & 167 & 166 & 149 & 136 & 136 \\ 128 & 136 & 123 & 136 & 154 & 180 & 198 & 154 \\ 123 & 105 & 110 & 149 & 136 & 136 & 180 & 166 \\ 110 & 136 & 123 & 123 & 123 & 136 & 154 & 136 \end{bmatrix}$$

Because the DCT is designed to work on pixel values ranging from -128 to 127, the original block is "leveled off" by subtracting 128 from each entry. This results in the following matrix.

$$M = \begin{bmatrix} 26 & -5 & -5 & -5 & -5 & -5 & -5 & 8 \\ 64 & 52 & 8 & 26 & 26 & 26 & 8 & -18 \\ 126 & 70 & 26 & 26 & 52 & 26 & -5 & -5 \\ 111 & 52 & 8 & 52 & 52 & 38 & -5 & -5 \\ 52 & 26 & 8 & 39 & 38 & 21 & 8 & 8 \\ 0 & 8 & -5 & 8 & 26 & 52 & 70 & 26 \\ -5 & -23 & -18 & 21 & 8 & 8 & 52 & 38 \\ -18 & 8 & -5 & -5 & -5 & 8 & 26 & 8 \end{bmatrix}$$

We are now ready to perform the Discrete Cosine Transform, which is accomplished by matrix multiplication.

$$D = TMT' \qquad 5$$

In Equation (5) matrix M is first multiplied on the left by the DCT matrix T from the previous section; this transforms the rows. The columns are then transformed by multiplying on the right by the transpose of the DCT matrix. This yields the following matrix.

$$D = \begin{bmatrix} 162.3 & 40.6 & 20.0 & 72.3 & 30.3 & 12.5 & -19.7 & -11.5 \\ 30.5 & 108.4 & 10.5 & 32.3 & 27.7 & -15.5 & 18.4 & -2.0 \\ -94.1 & -60.1 & 12.3 & -43.4 & -31.3 & 6.1 & -3.3 & 7.1 \\ -38.6 & -83.4 & -5.4 & -22.2 & -13.5 & 15.5 & -1.3 & 3.5 \\ -31.3 & 17.9 & -5.5 & -12.4 & 14.3 & -6.0 & 11.5 & -6.0 \\ -0.9 & -11.8 & 12.8 & 0.2 & 28.1 & 12.6 & 8.4 & 2.9 \\ 4.6 & -2.4 & 12.2 & 6.6 & -18.7 & -12.8 & 7.7 & 12.0 \\ -10.0 & 11.2 & 7.8 & -16.3 & 21.5 & 0.0 & 5.9 & 10.7 \end{bmatrix}$$

This block matrix now consists of 64 DCT coefficients, c_{ij}, where i and j range from 0 to 7. The top-left coefficient, c_{00}, correlates to the low frequencies of the original image block. As we move away from c_{00} in all directions, the DCT coefficients correlate to higher and higher frequencies of the image block, where c_{77} corresponds to the highest frequency. It is important to note that the human eye is most sensitive to low frequencies, and results from the quantization step will reflect this fact.

Quantization

Our 8x8 block of DCT coefficients is now ready for compression by quantization. A remarkable and highly useful feature of the JPEG process is that in this step, varying levels of image compression and quality are obtainable through selection of specific quantization matrices. This enables the user to decide on quality levels ranging from 1 to 100, where 1 gives the poorest image quality and highest compression, while 100 gives the best quality and lowest compression. As a result, the quality/compression ratio can be tailored to suit different needs.

Subjective experiments involving the human visual system have resulted in the JPEG standard quantization matrix. With a quality level of 50, this matrix renders both high compression and excellent decompressed image quality.

$$Q_{50} = \begin{bmatrix} 16 & 11 & 10 & 16 & 24 & 40 & 51 & 61 \\ 12 & 12 & 14 & 19 & 26 & 58 & 60 & 55 \\ 14 & 13 & 16 & 24 & 40 & 57 & 69 & 56 \\ 14 & 17 & 22 & 29 & 51 & 87 & 80 & 62 \\ 18 & 22 & 37 & 56 & 68 & 109 & 103 & 77 \\ 24 & 35 & 55 & 64 & 81 & 104 & 113 & 92 \\ 49 & 64 & 78 & 87 & 103 & 121 & 120 & 101 \\ 72 & 92 & 95 & 98 & 112 & 100 & 103 & 99 \end{bmatrix}$$

If, however, another level of quality and compression is desired, scalar multiples of the JPEG standard quantization matrix may be used. For a quality level greater than 50 (less compression, higher image quality), the standard quantization matrix is multiplied by (100-quality level)/50. For a quality level less than 50 (more compression, lower image quality), the standard quantization matrix is multiplied by 50/quality level. The scaled

quantization matrix is then rounded and clipped to have positive integer values ranging from 1 to 255. For example, the following quantization matrices yield quality levels of 10 and 90.

$$Q_{10} = \begin{bmatrix} 80 & 60 & 50 & 80 & 120 & 200 & 255 & 255 \\ 55 & 60 & 70 & 95 & 130 & 255 & 255 & 255 \\ 70 & 65 & 80 & 120 & 200 & 255 & 255 & 255 \\ 70 & 85 & 110 & 145 & 255 & 255 & 255 & 255 \\ 90 & 110 & 185 & 255 & 255 & 255 & 255 & 255 \\ 120 & 175 & 255 & 255 & 255 & 255 & 255 & 255 \\ 245 & 255 & 255 & 255 & 255 & 255 & 255 & 255 \\ 255 & 255 & 255 & 255 & 255 & 255 & 255 & 255 \end{bmatrix}$$

$$Q_{90} = \begin{bmatrix} 3 & 2 & 2 & 3 & 5 & 8 & 10 & 12 \\ 2 & 2 & 3 & 4 & 5 & 12 & 12 & 11 \\ 3 & 3 & 3 & 5 & 8 & 11 & 14 & 11 \\ 3 & 3 & 4 & 6 & 10 & 17 & 16 & 12 \\ 4 & 4 & 7 & 11 & 14 & 22 & 21 & 15 \\ 5 & 7 & 11 & 13 & 16 & 12 & 23 & 18 \\ 10 & 13 & 16 & 17 & 21 & 24 & 24 & 21 \\ 14 & 18 & 19 & 20 & 22 & 20 & 20 & 20 \end{bmatrix}$$

Quantization is achieved by dividing each element in the transformed image matrix D by the corresponding element in the quantization matrix, and then rounding to the nearest integer value. For the following step, quantization matrix Q_{50} is used.

$$C_{i,j} = round\left(\frac{D_{i,j}}{Q_{i,j}}\right) \qquad 6$$

$$C = \begin{bmatrix} 10 & 4 & 2 & 5 & 1 & 0 & 0 & 0 \\ 3 & 9 & 1 & 2 & 1 & 0 & 0 & 0 \\ -7 & -5 & 1 & -2 & -1 & 0 & 0 & 0 \\ -3 & -5 & 0 & -1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Recall that the coefficients situated near the upper-left corner correspond to the lower frequencies – to which the human eye is most sensitive – of the image block. In addition, the zeros represent the less important, higher frequencies that have been discarded, giving rise to

the lossy part of compression. As mentioned earlier, only the remaining nonzero coefficients will be used to reconstruct the image. It is also interesting to note the effect of different quantization matrices; use of Q_{10} would give C significantly more zeros, while Q_{90} would result in very few zeros.

Coding

The quantized matrix C is now ready for the final step of compression. Before storage, all coefficients of C are converted by an encoder to a stream of binary data (01101011...). In-depth coverage of the coding process is beyond the scope of this article. However, we can point out one key aspect that the reader is sure to appreciate. After quantization, it is quite common for most of the coefficients to equal zero. JPEG takes advantage of this by encoding quantized coefficients in the zig-zag sequence shown in Figure 1. The advantage lies in the consolidation of relatively large runs of zeros, which compress very well. The sequence in Figure 1 (4x4) continues for the entire 8x8 block.

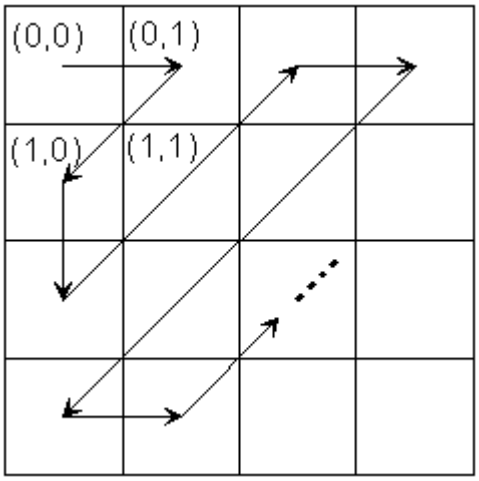

Figure 1

Decompression

Reconstruction of our image begins by decoding the bit stream representing the quantized matrix C. Each element of C is then multiplied by the corresponding element of the quantization matrix originally used.

$$R_{i,j} = Q_{i,j} \times C_{i,j} \qquad 7$$

$$R = \begin{bmatrix} 160 & 44 & 20 & 80 & 24 & 0 & 0 & 0 \\ 36 & 108 & 14 & 38 & 26 & 0 & 0 & 0 \\ -98 & -65 & 16 & -48 & -40 & 0 & 0 & 0 \\ -42 & -85 & 0 & -29 & 0 & 0 & 0 & 0 \\ -36 & 22 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The IDCT is next applied to matrix R, which is rounded to the nearest integer. Finally, 128 is added to each element of that result, giving us the decompressed JPEG version N of our original 8x8 image block M.

$$N = round(T' \; R \; T) + 128 \qquad 8$$

Comparison of Matrices

Let us now see how the JPEG version of our original pixel block compares.

$$Original = \begin{bmatrix} 154 & 123 & 123 & 123 & 123 & 123 & 123 & 136 \\ 192 & 180 & 136 & 154 & 154 & 154 & 136 & 110 \\ 254 & 198 & 154 & 154 & 180 & 154 & 123 & 123 \\ 239 & 180 & 136 & 180 & 180 & 166 & 123 & 123 \\ 180 & 154 & 136 & 167 & 166 & 149 & 136 & 136 \\ 128 & 136 & 123 & 136 & 154 & 180 & 198 & 154 \\ 123 & 105 & 110 & 149 & 136 & 136 & 180 & 166 \\ 110 & 136 & 123 & 123 & 123 & 136 & 154 & 136 \end{bmatrix}$$

$$Decompressed = \begin{bmatrix} 149 & 134 & 119 & 116 & 121 & 126 & 127 & 128 \\ 204 & 168 & 140 & 144 & 155 & 150 & 135 & 125 \\ 253 & 195 & 155 & 166 & 183 & 165 & 131 & 111 \\ 245 & 185 & 148 & 166 & 184 & 160 & 124 & 107 \\ 188 & 149 & 132 & 155 & 172 & 159 & 141 & 136 \\ 132 & 123 & 125 & 143 & 160 & 166 & 168 & 171 \\ 109 & 119 & 126 & 128 & 139 & 158 & 168 & 166 \\ 111 & 127 & 127 & 114 & 118 & 141 & 147 & 135 \end{bmatrix}$$

This is a remarkable result, considering that nearly 70% of the DCT coefficients were discarded prior to image block decompression/reconstruction. Given that similar results will occur with the rest of the blocks that constitute the entire image, it should be no surprise that

the JPEG image will be scarcely distinguishable from the original. Remember, there are 256 possible shades of gray in a black-and-white picture, and a difference of, say, 10, is barely noticeable to the human eye.

Pepper Example

We can do the DCT and quantization process on the peppers image.

Figure 2 – Peppers

Each eight by eight block is hit with the DCT, resulting in the image shown in Figure 3.

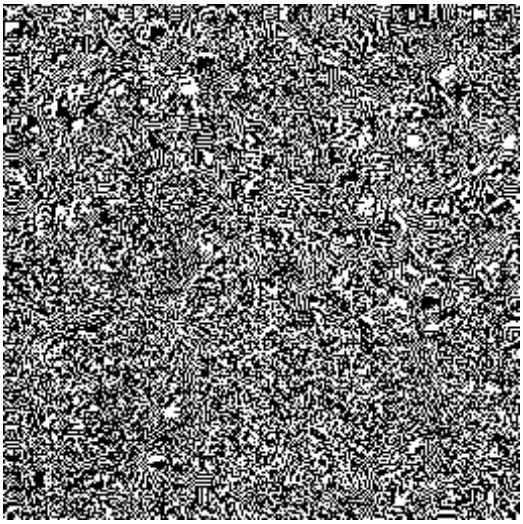

Figure 3 – DCT of Peppers

Each element in each block of the image is then quantized using a quantization matrix of quality level 50. At this point many of the elements become zeroed out, and the image takes up much less space to store.

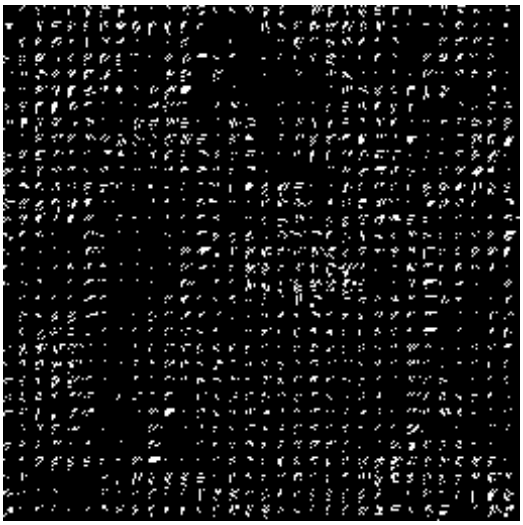

Figure 4 – Quantized DCT of Peppers

The image can now be decompressed using the inverse discrete cosine transform. At quality level 50 there is almost no visible loss in this image, but there is high compression. At lower quality levels, the quality goes down by a lot, but the compression does not increase very much.

Figure 5 – Original Peppers

Figure 6 – Quality 50 – 84% Zeros

Figure 7 – Quality 20 – 91% Zeros

Figure 8 – Quality 10 – 94% Zeros

More Examples

We can see what the compression does to other images. High contrast images, or images with a lot of high frequencies do not compress as well as smooth, low frequency images.

Figure 9 – Original

Figure 10 – Quality 15 – 90% Zeros

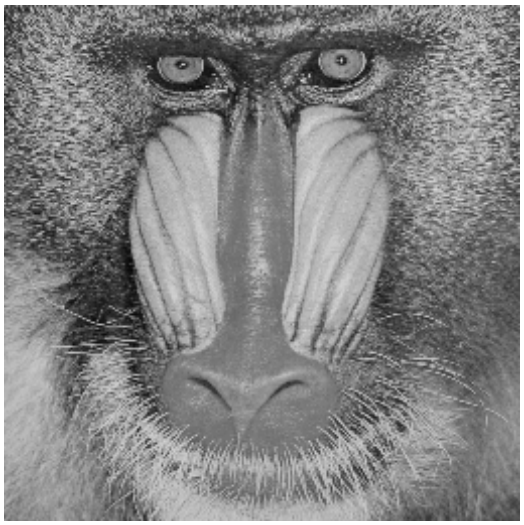

Figure 11 – Original

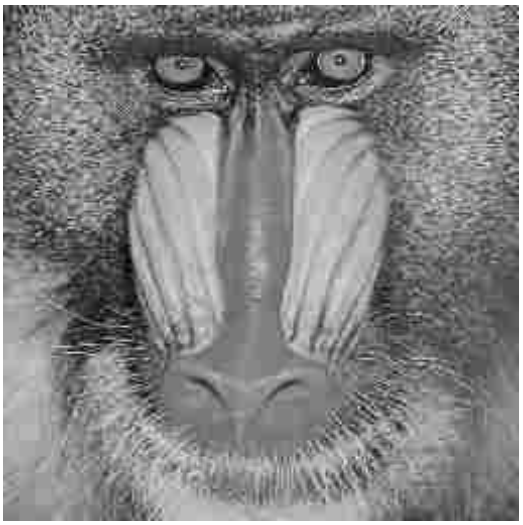

Figure 12 – Quality 15 – 88% Zeros

Bibliography

- Kesavan, Hareesh. *Choosing a DCT Quantization Matrix for JPEG Encoding*. Web page. http://www-ise.Stanford.EDU/class/ee392c/demos/kesavan/
- McGowan, John. *The Discrete Cosine Transform*. Web page. http://www.rahul.net/jfm/dct.html
- Wallace, Gregory K. *The JPEG Still Picture Compression Standard*. Paper submitted in December 1991 for publication in *IEEE Transactions on Consumer Electronics*.
- Wolfgang, Ray. *JPEG Tutorial*. Web page. http://www.imaging.org/tutorial/jpegtut1.html
- Our special thanks to David Arnold, math instructor extraordinaire.